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Tables: Two-Port Matrices

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Abstract

This short text is nothing more than a helpful collection of tables for two-port matrices. You can find tutorial texts in many places, for example in [1, chapter 15.3] (in English) or in [2, chapter 19] (in German). Most of the tables in this collection have been taken from [2, 3] and then extended.

I have done my best to remove all errors, but if you still can find one, please report it to the author: `schmid@isi.ee.ethz.ch`

References

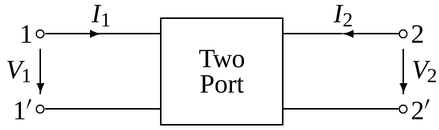
[1] Wai-Kai Chen, Ed., *The Circuits and Filters Handbook*, CRC Press, Inc., Boca Raton, Florida, 1995.

[2] G. Epprecht, *Technische Elektrizitätslehre III*, AMIV-Verlag, ETH Zürich, 3 edition, 1979.

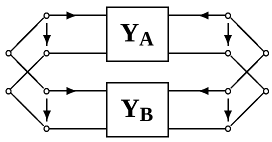
[3] Werner Bächtold, *Lineare Elemente der Höchstfrequenztechnik*, Verlag der Fachvereine, Zürich, 1994.

Contents

1 Definition of the four most important two-port matrices



1.1 Short circuit admittance matrix (admittance matrix)

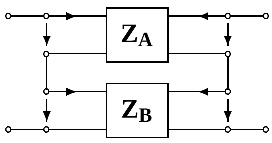


$$\mathbf{Y} = \mathbf{Y_A} + \mathbf{Y_B}$$

$$\mathbf{I} = \mathbf{Y} \cdot \mathbf{V}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

1.2 Open circuit impedance matrix (impedance matrix)

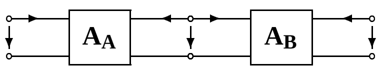


$$\mathbf{Z} = \mathbf{Z_A} + \mathbf{Z_B}$$

$$\mathbf{V} = \mathbf{Z} \cdot \mathbf{I}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

1.3 Transmission matrix (chain matrix)

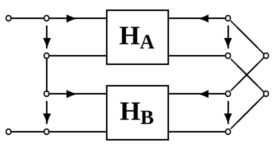


$$\mathbf{A} = \mathbf{A_A} \cdot \mathbf{A_B}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

1.4 Hybrid matrix



$$\mathbf{H} = \mathbf{H_A} + \mathbf{H_B}$$

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \mathbf{H} \cdot \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

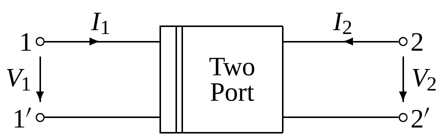
$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

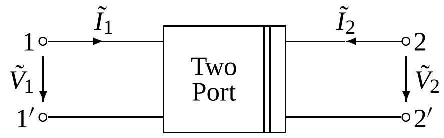
2 Conversions . . .

2.1 . . . between the two-port matrices

$\mathbf{Z}$	$\frac{1}{\det\mathbf{Y}}\begin{bmatrix} y_{22} & -y_{12} \\ -y_{21} & y_{11} \end{bmatrix}$	$\frac{1}{a_{21}}\begin{bmatrix} a_{11} & \det\mathbf{A} \\ 1 & a_{22} \end{bmatrix}$	$\frac{1}{h_{22}}\begin{bmatrix} \det\mathbf{H} & h_{12} \\ -h_{21} & 1 \end{bmatrix}$
$\frac{1}{\det\mathbf{Z}}\begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$	$\mathbf{Y}$	$\frac{1}{a_{12}}\begin{bmatrix} a_{22} & -\det\mathbf{A} \\ -1 & a_{11} \end{bmatrix}$	$\frac{1}{h_{11}}\begin{bmatrix} 1 & -h_{12} \\ h_{21} & \det\mathbf{H} \end{bmatrix}$
$\frac{1}{z_{21}}\begin{bmatrix} z_{11} & \det\mathbf{Z} \\ 1 & z_{22} \end{bmatrix}$	$-\frac{1}{y_{21}}\begin{bmatrix} y_{22} & 1 \\ \det\mathbf{Y} & y_{11} \end{bmatrix}$	$\mathbf{A}$	$-\frac{1}{h_{21}}\begin{bmatrix} \det\mathbf{H} & h_{11} \\ h_{22} & 1 \end{bmatrix}$
$\frac{1}{z_{22}}\begin{bmatrix} \det\mathbf{Z} & z_{12} \\ -z_{21} & 1 \end{bmatrix}$	$\frac{1}{y_{11}}\begin{bmatrix} 1 & -y_{12} \\ y_{21} & \det\mathbf{Y} \end{bmatrix}$	$\frac{1}{a_{22}}\begin{bmatrix} a_{12} & \det\mathbf{A} \\ -1 & a_{21} \end{bmatrix}$	$\mathbf{H}$

2.2 . . . when a two-port is reversed





$$\widetilde{\mathbf{Z}} = \begin{bmatrix} z_{22} & z_{21} \\ z_{12} & z_{11} \end{bmatrix} \qquad \widetilde{\mathbf{Y}} = \begin{bmatrix} y_{22} & y_{21} \\ y_{12} & y_{11} \end{bmatrix} \qquad \widetilde{\mathbf{A}} = \frac{1}{\det\mathbf{A}}\begin{bmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{bmatrix} \qquad \widetilde{\mathbf{H}} = \frac{1}{\det\mathbf{H}}\begin{bmatrix} h_{11} & -h_{21} \\ -h_{12} & h_{22} \end{bmatrix}$$

3 Conditions for Reciprocity and Symmetry

	symmetry	
$\mathbf{Z}$	$z_{12} = z_{21}$	$z_{11} = z_{22}$
$\mathbf{Y}$	$y_{12} = y_{21}$	$y_{11} = y_{22}$
$\mathbf{A}$	$\det\mathbf{A} = 1$	$a_{11} = a_{22}$
$\mathbf{H}$	$h_{12} = -h_{21}$	$\det\mathbf{H} = 1$
	reciprocity	

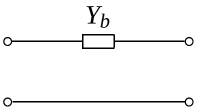
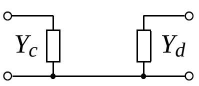
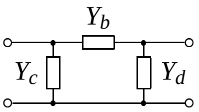
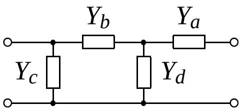
4 Input, Output and Transmission Functions

Function	open/short circuit				cond.	with a load impedance Z_L				①	②
$Z_{in} = \frac{V_1}{I_1}$	$\frac{1}{y_{11}}$	$\frac{\det \mathbf{Z}}{z_{22}}$	$\frac{a_{12}}{a_{22}}$	h_{11}	$(V_2 = 0)$	$\frac{y_{22} Z_L + 1}{\det \mathbf{Y}\, Z_L + y_{11}}$	$\frac{z_{11} Z_L + \det \mathbf{Z}}{Z_L + z_{22}}$	$\frac{a_{11} Z_L + a_{12}}{a_{21} Z_L + a_{22}}$	$\frac{\det \mathbf{H}\, Z_L + h_{11}}{h_{22} Z_L + 1}$	I_1	→
	$\frac{y_{22}}{\det \mathbf{Y}}$	z_{11}	$\frac{a_{11}}{a_{21}}$	$\frac{\det \mathbf{H}}{h_{22}}$	$(I_2 = 0)$						→
$Z_{out} = \frac{V_2}{I_2}$	$\frac{1}{y_{22}}$	$\frac{\det \mathbf{Z}}{z_{11}}$	$\frac{a_{12}}{a_{11}}$	$\frac{h_{11}}{\det \mathbf{H}}$	$(V_1 = 0)$	$\frac{y_{11} Z_L + 1}{\det \mathbf{Y}\, Z_L + y_{22}}$	$\frac{z_{22} Z_L + \det \mathbf{Z}}{Z_L + z_{11}}$	$\frac{a_{22} Z_L + a_{12}}{a_{21} Z_L + a_{11}}$	$\frac{Z_L + h_{11}}{h_{22} Z_L + \det \mathbf{H}}$	I_2	←
	$\frac{y_{11}}{\det \mathbf{Y}}$	z_{22}	$\frac{a_{22}}{a_{21}}$	$\frac{1}{h_{22}}$	$(I_1 = 0)$						←
$Z_{fwd} = \frac{V_2}{I_1}$	$-\frac{y_{21}}{\det \mathbf{Y}}$	z_{21}	$\frac{1}{a_{21}}$	$-\frac{h_{21}}{h_{22}}$	$(I_2 = 0)$	$\frac{-y_{21} Z_L}{\det \mathbf{Y}\, Z_L + y_{11}}$	$\frac{z_{21} Z_L}{Z_L + z_{22}}$	$\frac{Z_L}{a_{21} Z_L + a_{22}}$	$\frac{-h_{21} Z_L}{h_{22} Z_L + 1}$	I_1	→
$Z_{rev} = \frac{V_1}{I_2}$	$-\frac{y_{12}}{\det \mathbf{Y}}$	z_{12}	$\frac{\det \mathbf{A}}{a_{21}}$	$\frac{h_{12}}{h_{22}}$	$(I_1 = 0)$	$\frac{-y_{12} Z_L}{\det \mathbf{Y}\, Z_L + y_{22}}$	$\frac{z_{12} Z_L}{Z_L + z_{11}}$	$\frac{\det \mathbf{A}\, Z_L}{a_{21} Z_L + a_{11}}$	$\frac{h_{12} Z_L}{h_{22} Z_L + \det \mathbf{H}}$	I_2	←
$Y_{fwd} = \frac{I_2}{V_1}$	y_{21}	$-\frac{z_{21}}{\det \mathbf{Z}}$	$-\frac{1}{a_{12}}$	$\frac{h_{21}}{h_{11}}$	$(V_2 = 0)$	$\frac{y_{21}}{y_{22} Z_L + 1}$	$\frac{-z_{21}}{z_{11} Z_L + \det \mathbf{Z}}$	$\frac{-1}{a_{11} Z_L + a_{12}}$	$\frac{h_{21}}{\det \mathbf{H}\, Z_L + h_{11}}$	V_1	→
$Y_{rev} = \frac{I_1}{V_2}$	y_{12}	$-\frac{z_{12}}{\det \mathbf{Z}}$	$-\frac{\det \mathbf{A}}{a_{12}}$	$-\frac{h_{12}}{h_{11}}$	$(V_1 = 0)$	$\frac{y_{12}}{y_{11} Z_L + 1}$	$\frac{-z_{12}}{z_{22} Z_L + \det \mathbf{Z}}$	$\frac{-\det \mathbf{A}}{a_{22} Z_L + a_{12}}$	$\frac{-h_{12}}{Z_L + h_{11}}$	V_2	←
$T_{v,fwd} = \frac{V_2}{V_1}$	$-\frac{y_{21}}{y_{22}}$	$\frac{z_{21}}{z_{11}}$	$\frac{1}{a_{11}}$	$-\frac{h_{21}}{\det \mathbf{H}}$	$(I_2 = 0)$	$\frac{-y_{21} Z_L}{y_{22} Z_L + 1}$	$\frac{z_{21} Z_L}{z_{11} Z_L + \det \mathbf{Z}}$	$\frac{Z_L}{a_{11} Z_L + a_{12}}$	$\frac{-h_{21} Z_L}{\det \mathbf{H}\, Z_L + h_{11}}$	V_1	→
$T_{v,rev} = \frac{V_1}{V_2}$	$-\frac{y_{12}}{y_{11}}$	$-\frac{z_{12}}{z_{22}}$	$\frac{\det \mathbf{A}}{a_{22}}$	h_{12}	$(I_1 = 0)$	$\frac{-y_{12} Z_L}{y_{11} Z_L + 1}$	$\frac{z_{12} Z_L}{z_{22} Z_L + \det \mathbf{Z}}$	$\frac{\det \mathbf{A}\, Z_L}{a_{22} Z_L + a_{12}}$	$\frac{h_{12} Z_L}{Z_L + h_{11}}$	V_2	←
$T_{c,fwd} = -\frac{I_2}{I_1}$	$-\frac{y_{21}}{y_{11}}$	$\frac{z_{21}}{z_{22}}$	$\frac{1}{a_{22}}$	$-h_{21}$	$(V_2 = 0)$	$\frac{-y_{21}}{\det \mathbf{Y}\, Z_L + y_{11}}$	$\frac{z_{21}}{Z_L + z_{22}}$	$\frac{1}{a_{21} Z_L + a_{22}}$	$\frac{-h_{21}}{h_{22} Z_L + 1}$	I_1	→
$T_{c,rev} = -\frac{I_1}{I_2}$	$-\frac{y_{12}}{y_{22}}$	$\frac{z_{12}}{z_{11}}$	$\frac{\det \mathbf{A}}{a_{11}}$	$\frac{h_{12}}{\det \mathbf{H}}$	$(V_1 = 0)$	$\frac{-y_{12}}{\det \mathbf{Y}\, Z_L + y_{22}}$	$\frac{z_{12}}{Z_L + z_{11}}$	$\frac{\det \mathbf{A}}{a_{21} Z_L + a_{11}}$	$\frac{h_{12}}{h_{22} Z_L + \det \mathbf{H}}$	I_2	←

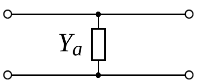
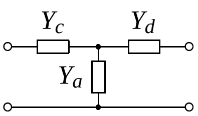
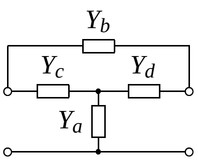
Column ① indicates the source driving the two-port, column ② indicates the direction in which the two-port is driven.

5 Twoport Matrix Examples

5.1 Circuits Based on the π-Network

	Y	**A**
	$Y_b\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$	$\begin{bmatrix}1 & \frac{1}{Y_b}\\ 0 & 1\end{bmatrix}$
	$\begin{bmatrix}Y_c & 0\\ 0 & Y_d\end{bmatrix}$	does not exist
	$\begin{bmatrix}Y_b+Y_c & -Y_b\\ -Y_b & Y_b+Y_d\end{bmatrix}$	$\frac{1}{Y_b}\begin{bmatrix}Y_b+Y_d & 1\\ Y_bY_c+Y_bY_d+Y_cY_d & Y_b+Y_c\end{bmatrix}$
	$\frac{1}{Y_a+Y_b+Y_d}\begin{bmatrix}(Y_b+Y_c)(Y_a+Y_d)+Y_bY_c & -Y_aY_b\\ -Y_aY_b & Y_a(Y_b+Y_d)\end{bmatrix}$	$\frac{1}{Y_aY_b}\begin{bmatrix}Y_a(Y_b+Y_d) & Y_a+Y_b+Y_d\\ Y_a(Y_bY_c+Y_bY_d+Y_cY_d) & (Y_b+Y_c)(Y_a+Y_d)+Y_bY_c\end{bmatrix}$

5.2 Circuits Based on the T-Network

	Y	**A**
	does not exist	$\begin{bmatrix}1 & 0\\ Y_a & 1\end{bmatrix}$
	$\frac{1}{Y_a+Y_c+Y_d}\begin{bmatrix}Y_c(Y_a+Y_d) & -Y_cY_d\\ -Y_cY_d & Y_d(Y_a+Y_c)\end{bmatrix}$	$\frac{1}{Y_cY_d}\begin{bmatrix}Y_d(Y_a+Y_c) & Y_a+Y_c+Y_d\\ Y_aY_cY_d & Y_c(Y_a+Y_d)\end{bmatrix}$
	$\frac{1}{Y_a+Y_c+Y_d}\begin{bmatrix}(Y_c+Y_b)(Y_a+Y_d)+Y_bY_c & -Y_b(Y_a+Y_c+Y_d)-Y_cY_d\\ -Y_b(Y_a+Y_c+Y_d)-Y_cY_d & (Y_a+Y_c)(Y_b+Y_d)+Y_bY_d\end{bmatrix}$	$\frac{1}{Y_b(Y_a+Y_c+Y_d)+Y_cY_d}\begin{bmatrix}(Y_a+Y_c)(Y_b+Y_d)+Y_bY_d & Y_a+Y_c+Y_d\\ Y_a(Y_bY_c+Y_bY_d+Y_cY_d) & (Y_b+Y_c)(Y_a+Y_d)+Y_bY_c\end{bmatrix}$

6 Miscellaneous tables

6.1 Twoport matrix determinants

$\det \mathbf{Y} =$	$y_{11}y_{22} - y_{12}y_{21}$	$1/\det \mathbf{Z}$	a_{21}/a_{12}	h_{22}/h_{11}
$\det \mathbf{Z} =$	$1/\det \mathbf{Y}$	$z_{11}z_{22} - z_{12}z_{21}$	a_{12}/a_{21}	h_{11}/h_{22}
$\det \mathbf{A} =$	y_{12}/y_{21}	z_{12}/z_{21}	$a_{11}a_{22} - a_{12}a_{21}$	$-h_{12}/h_{21}$
$\det \mathbf{H} =$	y_{22}/y_{11}	z_{11}/z_{22}	a_{11}/a_{22}	$h_{11}h_{22} - h_{12}h_{21}$

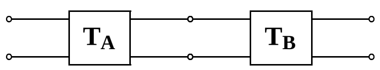
6.2 Scattering Matrices

Scattering Matrix

$$\mathbf{B} = \mathbf{S} \cdot \mathbf{A}$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Transfer Matrix (wave chain matrix)



$$\mathbf{T} = \mathbf{T_A} \cdot \mathbf{T_B}$$

$$\begin{bmatrix} b_1 \\ a_1 \end{bmatrix} = \mathbf{T} \cdot \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} b_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \cdot \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

Conversion

$$\mathbf{S} = \frac{1}{t_{22}} \begin{bmatrix} t_{12} & \det \mathbf{T} \\ 1 & -t_{21} \end{bmatrix} \qquad \mathbf{T} = \frac{1}{s_{21}} \begin{bmatrix} -\det \mathbf{S} & s_{11} \\ -s_{22} & 1 \end{bmatrix}$$

(Formulas for the conversion from and to the other two-port matrices will be added soon.)