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Two curious integrals and a graphic proof

Hanspeter Schmid

Hanspeter Schmid is the professor for analog microelectronics at the Institute of Microelectronics (IME) of Fachhochschule Nordwestschweiz (FHNW), with interests in sensor electronics, signal integrity, and sigma-delta conversion. In 2011–2012 he was a distinguished lecturer of the IEEE CAS Society.

1 Introduction

This paper deals with some integrals of products of the sinc function defined as

$$\operatorname{sinc}(t) = \begin{cases} \sin(t)/t & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases} . \tag{1}$$

2001 veröffentlichten David und Jonathan Borwein im Ramanujan Journal eine Reihe von Integralformeln, die dem Namensgeber der Zeitschrift sicher auch gefallen hätte:

$$\int_0^\infty \prod_{k=0}^{n} \operatorname{sinc}\left(\frac{t}{2k+1}\right) dt = \frac{\pi}{2}$$

für $n = 0, 1, \ldots, 6$. Wer nun gewettet hätte, das ginge so weiter, hätte verloren: Der Wert des Integrals für $n = 7$ liegt nämlich mit

$$\frac{467807924713440738696537864469}{935615849440640907310521750000}\,\pi$$

haarscharf unterhalb von $\frac{\pi}{2}$ und fällt für wachsende n weiter. Der Beweis der Borweins lieferte leider wenig Einsicht in das Phänomen. In der vorliegenden Arbeit führt der Autor ein sehr einfaches und intuitives Argument ins Feld, welches den Effekt erklärt. Darüber hinaus wird ein modifiziertes Beispiel präsentiert, wo der Wert $\frac{\pi}{2}$ erst nach 57 Schritten unterboten wird.

The Borweins described an astonishing fact using these sinc functions [1, 2]:

$$\int_0^\infty \operatorname{sinc}(t)\, dt = \frac{\pi}{2} \tag{2}$$

$$\int_0^\infty \operatorname{sinc}(t) \cdot \operatorname{sinc}\left(\frac{t}{3}\right) dt = \frac{\pi}{2} \tag{3}$$

$$\int_0^\infty \operatorname{sinc}(t) \cdot \operatorname{sinc}\left(\frac{t}{3}\right) \cdot \operatorname{sinc}\left(\frac{t}{5}\right) dt = \frac{\pi}{2} \tag{4}$$

$$\vdots$$

$$\int_0^\infty \operatorname{sinc}(t) \cdot \operatorname{sinc}\left(\frac{t}{3}\right) \cdot \ldots \cdot \operatorname{sinc}\left(\frac{t}{13}\right) dt = \frac{\pi}{2} \tag{5}$$

but then

$$\int_0^\infty \operatorname{sinc}(t) \cdot \operatorname{sinc}\left(\frac{t}{3}\right) \cdot \ldots \cdot \operatorname{sinc}\left(\frac{t}{13}\right) \cdot \operatorname{sinc}\left(\frac{t}{15}\right) dt < \frac{\pi}{2}\,. \tag{6}$$

The value of that last integral is $\approx 0.499999999992646\,\pi$.

As they write, *when this fact was recently verified by a researcher using a computer algebra package, he concluded that there must be a "bug" in the software.* It is not a bug, though; this series of integrals *really* only results in $\pi/2$ up to a certain point, and then breaks down. This astonishes most mathematically educated readers, as especially those readers mentally extrapolate the sequence shown above and find it surprising that something fundamental should change when the factor $\operatorname{sinc}(x/15)$ is introduced in (6).

This was proven in [1], but the proof is not graphic, and while it is intellectually appealing, it is difficult to really understand. In this paper we provide a simpler version of the Borweins' proof which gives a graphic and therefore intuitive understanding of *what* it is that changes fundamentally when the sequence breaks down.

In addition, it also lets us show that the integral series above breaks down much later if there is another factor $2\cos(t)$ in the integral:

$$\int_0^\infty 2\cos(t) \cdot \operatorname{sinc}(t)\, dt = \frac{\pi}{2} \tag{7}$$

$$\vdots$$

$$\int_0^\infty 2\cos(t) \cdot \operatorname{sinc}(t) \cdot \operatorname{sinc}\left(\frac{t}{3}\right) \cdot \ldots \cdot \operatorname{sinc}\left(\frac{t}{111}\right) dt = \frac{\pi}{2} \tag{8}$$

$$\int_0^\infty 2\cos(t) \cdot \operatorname{sinc}(t) \cdot \operatorname{sinc}\left(\frac{t}{3}\right) \cdot \ldots \cdot \operatorname{sinc}\left(\frac{t}{111}\right) \cdot \operatorname{sinc}\left(\frac{t}{113}\right) dt < \frac{\pi}{2}\,. \tag{9}$$

We will now show where, and why, these two series break down, and we will do this in two steps: in Section 2, we show mostly graphically what happens when a rectangle function is repeatedly convolved with narrower unit-area rectangles. In Section 3, this insight then leads to a comparatively simple calculation of the point where the above series break down.

Correction: on this page, all occurrences of G(η−ω) should actually be G(ω−η).

2 Convolution with a rectangle

The convolution of two functions is defined as [3, 4]

$$F(\omega) * G(\omega) = \int_{\eta=-\infty}^{\infty} F(\eta) G(\eta - \omega) d\eta \, . \tag{10}$$

If G is a unit-area rectangle of width $1/k$, e.g.,

$$G(\omega) = k \operatorname{rect}(k\omega) \ ,$$

then this convolution operation corresponds to taking the moving average of the function F, where the width of the moving-average window is the width of the unit rectangle. Figure 1 shows examples for F and G for three different values of ω. In this example, the function F has a plateau, i.e., it is constant $F(0)$ in the middle, has two falling slopes towards the outside, and is zero outside these slopes.

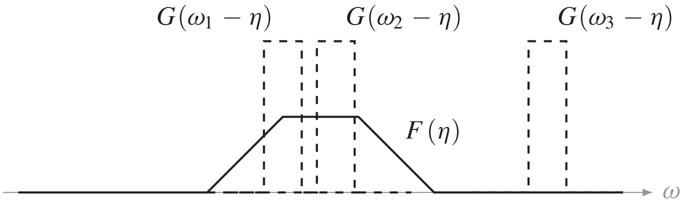

Fig. 1 Factors in the convolution integral (10) for three different values of ω.

For ω_3, the product $F(\eta)G(\eta - \omega)$ is obviously zero for all η and $F * G = 0$ at ω_3. This is the case for all ω such that G lies outside the region where F is non-zero. For ω_2, the product $F(\eta)G(\eta - \omega)$ is a rectangle with area $F(0)$, hence $F * G = F(0)$ at ω_2. This is the case for all ω such that G lies within the region where $F = F(0)$. For ω_1, the product $F(\eta)G(\eta - \omega)$ is a shape with finite area smaller than $F(0)$. So $0 < F * G < F(0)$ for ω_1 and all similar cases. And this is already everything we need.

Figure 2 shows two rectangles with unit area and widths 1 and $\frac{1}{2}$. We will now discuss, graphically first, what happens when F is repeatedly convolved with G.

Let $F(\omega) = \operatorname{rect}(\omega)$ and $G(\omega) = 2 \operatorname{rect}(2\omega)$, as shown in Figure 2. Now we look at a series of convolution products shown in Figure 3. There, $F_0 = F$, $F_1 = F_0 * G$,

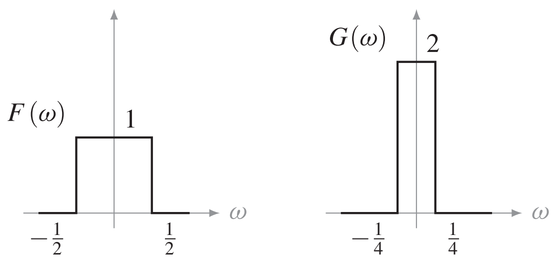

Fig. 2 Two unit-area rectangles F and G with different width.

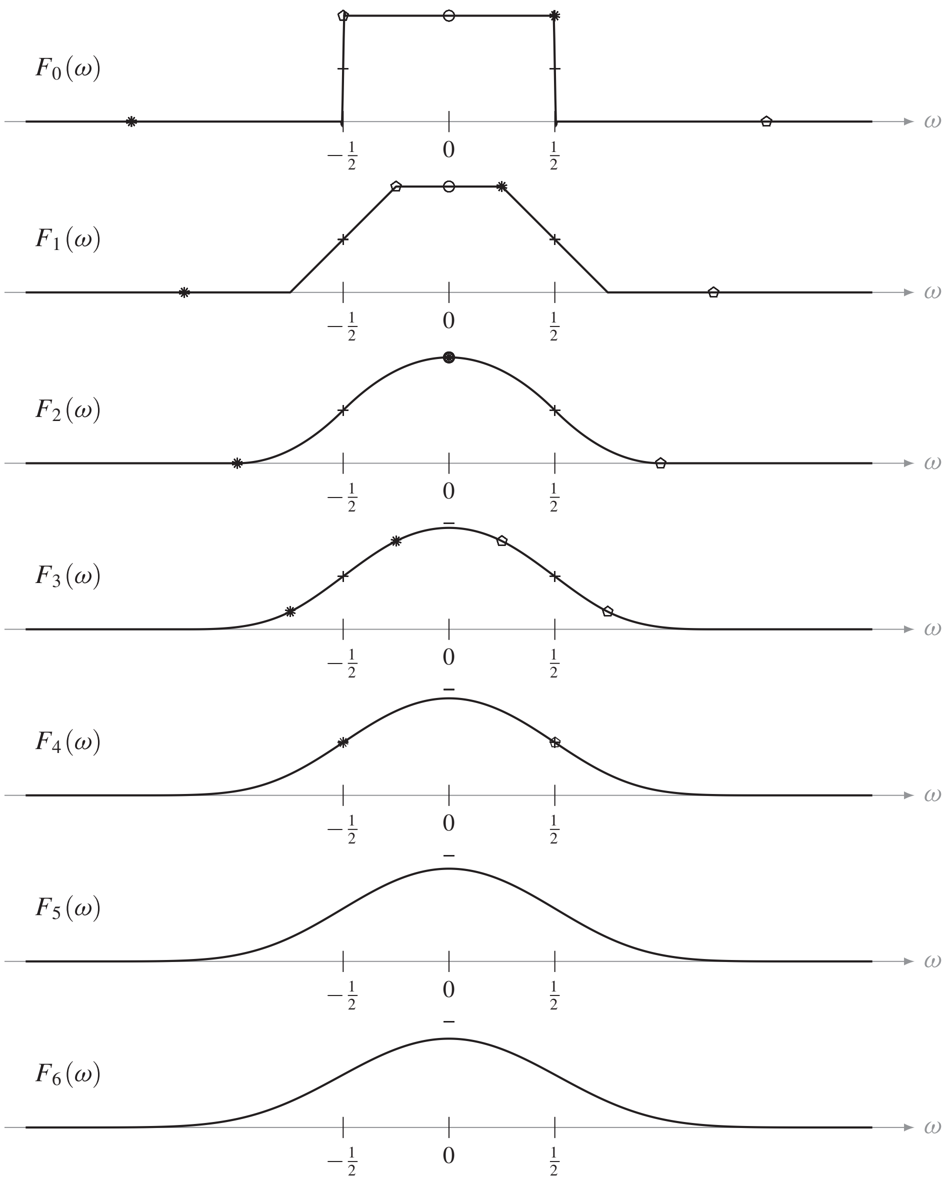

Fig. 3 Convolution products: every F_k is the function above convolved with G, a unit-area rectangle of width $\frac{1}{2}$.

$F_2 = F_1 * G$, and so on. All these functions have markers on them that help us see what happens qualitatively.

The first graph shows F_0 with four types of markers: a circle, ○, marking the point where $F_0(0) = 1$; two crosses, +, marking the two points where $F_0\big(\pm\frac{1}{2}\big) = \frac{1}{2}$, which is how the rectangle function rect(·) is conventionally defined; a ten-pointed star, ❋, marking the begin and end of the range in which the left slope of F_0 is point-symmetric around $\left(-\frac{1}{2}, \frac{1}{2}\right)$; and a pentagon, ⬠, marking the begin and end of the range in which the right slope of F_0 is point-symmetric around $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Intuitively seen, repeated convolution with G is a step-wise erosion process. In this erosion process, each convolution step reduces the width of the plateau as well as the width of the point-symmetric regions by twice the width of G, symmetrically. Proceeding through Figure 3 from top to bottom, it can be seen how the plateau is eroded from F_0 to F_1 and becomes a single point in F_2, and how it disappears from F_3 onwards, such that $F_{3,4,\ldots}(0) < 1$. F_3 still has point-symmetric regions, though. These point-symmetric regions are reduced to single points in F_4, and are then eroded by the repeated convolution from F_5 onwards, such that $F_{5,6,\ldots}\left(\pm\frac{1}{2}\right) < \frac{1}{2}$.

All that remains to do at this point is a generalization of this discussion, and this goes as follows: Let $\{a_k\}$ be a monotonically non-increasing series of positive real numbers. Let $F_0 = a_0 \operatorname{rect}(a_0\omega)$ and recursively define $F_k = F_{k-1} * a_k \operatorname{rect}(a_k\omega)$. The function F_k then has the total width $\sum_{k=0}^{n} a_k$. The plateau gets eroded when the sum of the widths of the rectangles convolved with F_0 is greater than the width of F_0:

$$
\begin{aligned}
&F_0(0) = 1 \\
&F_n(0) = 1 \quad \text{for all } n \text{ such that } \sum_{k=1}^{n} a_k \le a_0 \\
&F_n(0) < F_{n-1}(0) \quad \text{otherwise.}
\end{aligned}
$$

The erosion of the symmetry points happens when that sum is twice as large:

$$
\begin{aligned}
&F_0\left(\frac{a_0}{2}\right) = F_0\left(-\frac{a_0}{2}\right) = \frac{1}{2} \\
&F_n\left(\frac{a_0}{2}\right) = F_n\left(-\frac{a_0}{2}\right) = \frac{1}{2} \quad \text{for all } n \text{ such that } \sum_{k=1}^{n} a_k \le 2a_0 \\
&F_n\left(\frac{a_0}{2}\right) = F_n\left(-\frac{a_0}{2}\right) < F_{n-1}\left(\frac{a_0}{2}\right) \quad \text{otherwise.}
\end{aligned}
$$

3 Calculating the two curious integrals

Now we have enough to tackle (2)–(9). The arguments of those integrals are even functions, so in general, we can calculate

$$
2\tau_n = \int_{-\infty}^{\infty} \prod_{k=0}^{n} \operatorname{sinc}(a_k t)\, dt \tag{11}
$$

and

$$
2\varepsilon_n = \int_{-\infty}^{\infty} 2\cos a_0 t \prod_{k=0}^{n} \operatorname{sinc}(a_k t)\, dt \ . \tag{12}
$$

In order to do this, we use the Fourier transform in the form used in engineering and signal processing [3, 4],

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \; ; \tag{13}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \; . \tag{14}$$

It follows from (13) that

$$F(0) = \int_{-\infty}^{\infty} f(t)dt \; , \tag{15}$$

hence the integral (11) can be calculated by evaluating the Fourier transform of the integral's argument at $\omega = 0$.

In general, the Fourier transform of a product of functions is

$$\int_{-\infty}^{\infty} f(t)g(t)e^{-j\omega t}dt = \frac{1}{2\pi}F(\omega) * G(\omega)\, . \tag{16}$$

The Fourier transform of the sinc function is

$$\int_{-\infty}^{\infty} \operatorname{sinc}(a_k t)e^{-j\omega t}dt = \frac{2\pi}{2a_k}\operatorname{rect}\left(\frac{\omega}{2a_k}\right)\; . \tag{17}$$

If we use the commutativity of the convolution, (16) can be written as

$$\int_{-\infty}^{\infty} f(t)g(t)e^{-j\omega t}dt = F(\omega) * \frac{1}{2\pi}G(\omega)\, . \tag{18}$$

If $g(t) = \operatorname{sinc}(a_k t)$, then the second term of the convolution in (18) is a unit-area rectangle of width $2a_k$.

We now see two things: First, the Fourier transform of the product of sinc functions in (11) is simply a series of convolutions of the function $\frac{\pi}{a_0}\operatorname{rect}\big(\frac{\omega}{2a_0}\big)$, which is a rectangle of width $2a_0$ and height $\frac{\pi}{a_0}$, with unit-area rectangles of width $2a_1$, $2a_2$, and so on. Second, the value of $2\tau_n$ from (11) is the value of that Fourier transform at $\omega = 0$, which starts to decrease when the plateau of $\frac{\pi}{a_0}\operatorname{rect}\big(\frac{\omega}{2a_0}\big)$ is eroded. This is very similar to the graphic discussion in Section 2. It therefore follows immediately that

$$\begin{aligned} 2\tau_0 &= \frac{\pi}{a_0} \\ 2\tau_n &= \frac{\pi}{a_0} \quad \text{for all } n \text{ such that } \sum_{k=1}^{n} a_k \le a_0 \\ 2\tau_n &< 2\tau_{n-1} \quad \text{otherwise.} \end{aligned}$$

Therefore the integral series (2)–(6) breaks down when the sum $a_1 + \cdots + a_n$ exceeds 1, which is when the term $\operatorname{sinc}(x/15)$ comes into the product. This is what the Borweins proved in [1].

Now on to (12). We first replace the $\cos(\cdot)$ by its exponential representation and then use the linearity of the integration to obtain a sum of integrals:

$$2\varepsilon_n = \int_{-\infty}^{\infty} 2\cos a_0 t \prod_{k=0}^{n} \operatorname{sinc}(a_k t)dt = \int_{-\infty}^{\infty} \left(e^{ja_0 t} + e^{-ja_0 t}\right) \prod_{k=0}^{n} \operatorname{sinc}(a_k t)dt$$
$$= \int_{-\infty}^{\infty} e^{ja_0 t} \prod_{k=0}^{n} \operatorname{sinc}(a_k t)dt + \int_{-\infty}^{\infty} e^{-ja_0 t} \prod_{k=0}^{n} \operatorname{sinc}(a_k t)dt \ . \tag{19}$$

This additional exponential factor has a distinct effect in the Fourier transform (13). It causes a frequency shift:

$$\int_{-\infty}^{\infty} e^{-ja_0 t} f(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} f(t) e^{-j(\omega + a_0)t} dt \ .$$

Therefore the value of the second integral in (19) is the value of the Fourier transform of the product of sinc functions not for $\omega = 0$, as before, but for $\omega = -a_0$. And this is precisely where we find the symmetry point of one slope in Section 2. Similarly, the first integral in (19) has the value at the other slope's symmetry point, at $\omega = +a_0$. Without further effort, we see that the series in (12) breaks down when the symmetry points of the slopes are eroded, and this gives

$$2\varepsilon_0 = \frac{\pi}{a_0}, \quad 2\varepsilon_n = \frac{\pi}{a_0} \quad \text{for all } n \text{ such that } \sum_{k=1}^{n} a_k \le 2a_0$$
$$2\varepsilon_n < 2\varepsilon_{n-1} \quad \text{otherwise.}$$

Therefore, the integral series (7)–(9) breaks down when the sum $a_1 + \cdots + a_n$ exceeds 2, which is when the term $\operatorname{sinc}(x/113)$ comes into the product, much later than for the first series.

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Hanspeter Schmid
Fachhochschule Nordwestschweiz (FHNW)
Institut für Mikroelektronik (IME)
Steinackerstrasse 1
CH-5210 Windisch, Schweiz
hanspeter.schmid@fhnw.ch