

Efficient simulation of harmonic distortion in discrete-time circuits

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Abstract—Espley showed in 1933 how to calculate harmonic distortion (HD) and inter-modulation distortion (IMD) using few equidistant points on the transfer characteristic of a circuit. This method was applied later to derive simplified formulae for HD in continuous-time amplifiers. It is shown in this paper that the main strength of the method lies in HD simulations in discrete-time amplifiers and filters. To show this, the theory is presented in a simple way and extended to distortion-induced offset. The precision for different HD levels is discussed for the first time in literature, showing that the method is precise to within 0.1 dB for distortions of -30 dB and below. Application examples are given to show that simulating with this method results in simulation times that are half as long as with the shortest possible (and barely practical) simulations using discrete Fourier transforms.

I. INTRODUCTION

The idea of using few equidistant points for calculating harmonic distortion is very old, even the 1933/34 publications by Espley [1], [2] describe one of the formulae as “well known”. He used the method to calculate harmonic distortions for a given input signal from a transfer characteristic of a vacuum tube given graphically in a data sheet.

Espley’s method was discussed in a few follow-up papers and was soon adapted to other problems, e.g., the inverse problem, where the transfer characteristic is estimated from measured harmonic-distortion values [3]. With the advent of computers, the method was all but forgotten: very few engineers now try to extract harmonic distortion information from transfer characteristics by ruler and pencil.

There are, however, two applications left even in modern times where the equidistant-point method can be used: one is the approximated calculation of harmonic distortion in circuits, as shown for current mirrors in [4] and for class-AB stages in [5]; the other is to speed up the simulation of harmonic distortion, as first shown for continuous-time filters in [6]. In spite of the age of the method, there are still many questions left unanswered by the literature that we answer with this paper:

What is the error of using an equidistant-point approximation? Less than 0.1 dB for harmonic distortions (HDs) of -30 dB or below. — *What is the minimum number of points needed for the calculation or simulation of n -th order harmonic distortion?* The first odd number greater than n is sufficient for HDs ≤ -30 dB. — *Would non-equidistant points give better results?* Not for HDs ≤ -30 dB. — *How does the equidistant-point method compare to using a discrete Fourier transform (DFT)?* As a rule of thumb, a $2(n-1)$ -point-DFT and an n -equidistant-point calculation give similar errors.

To support the above answers to the questions, the method

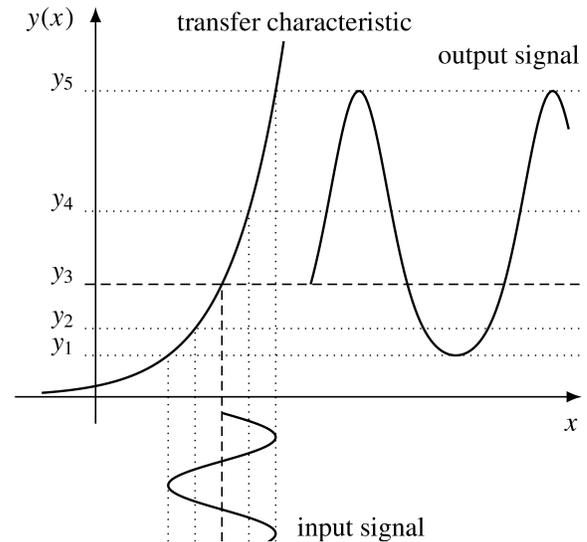


Fig. 1. Transfer characteristic, input and output signal, and five equidistant points on the characteristic ($n = 5$).

is explained in Sec. II and numerical comparisons are made in Sec. III. Section IV then briefly shows how the method can be applied, and provides a real-world example in which the simulation time of 5.4 hours used by the standard method is reduced to 30 minutes. The application to simplified calculations is shown in Sec. V, and then the strengths and limits are discussed in Sec. VI.

II. CALCULATION OF OFFSET AND DISTORTION FROM EQUIDISTANT POINTS ON THE TRANSFER CHARACTERISTIC

Fig. 1 shows the transfer characteristic $y = \frac{1}{5}e^{x+2}$ of some block (chosen because it looks nice) together with a sinusoidal input signal of magnitude 1, a distorted output signal, and x equidistant points on $y(x)$. Espley showed in [1] how the harmonic distortion can be approximated by fitting a polynomial through the equidistant points and then calculating the harmonics caused by this polynomial.

The calculation can best be shown with only three equidistant points, y_1, y_2, y_3 , where y_1 is the output for the most negative input $x = -e$, y_2 for the medium input $x = 0$, and y_3 for the most positive input $x = +e$.¹

¹We can choose these values without loss of generality because $x = 0$ for the medium value can always be achieved with a coordinate transform.

The polynomial we use is $y = a_0 + a_1x + a_2x^2$, so we will get:

$$y_1 = a_0 - a_1e + a_2e^2, \quad y_2 = a_0, \quad y_3 = a_0 + a_1e + a_2e^2. \quad (1)$$

This equation system can be solved for the polynomial coefficients:

$$a_0 = y_2, \quad a_1 = -\frac{y_1 - y_3}{2e}, \quad a_2 = -\frac{-y_1 + 2y_2 - y_3}{2e^2}. \quad (2)$$

And now we choose $x = e \sin \omega t$ as the input signal and calculate the polynomial by inserting the coefficients and the signal into $y = a_0 + a_1x + a_2x^2$. This gives:

$$\frac{(y_1 + 2y_2 + y_3) - 2(y_1 - y_3) \sin \omega t - (y_1 - 2y_2 + y_3) \cos 2\omega t}{4} \quad (3)$$

The e cancels out and we can now directly see the output values at DC, at ωt and at $2\omega t$. This allows us to compute the DC offset caused by harmonic distortion and the magnitude of the second harmonic relative to the first harmonic:

$$\text{Offset} = \frac{1}{4}(y_1 + 2y_2 + y_3) - y_2 = \frac{y_1 - 2y_2 + y_3}{4}, \quad (4)$$

$$H_2 = \frac{y_1 - 2y_2 + y_3}{2(y_1 - y_3)}. \quad (5)$$

The same calculation can be carried out for different numbers of equidistant points on the transfer characteristic and for other harmonics i ; the H_i for $n = 3, 5, 7, 9$ are given in the Appendix.

For the case $n = 5$ we have also calculated the third-order inter-modulation distortion (IM3) by using an input signal $x = e \sin \omega_1 t + e \sin \omega_2 t$. We did this only for $n = 5$ because this formula will be sufficient for $\text{IM3} \leq -30$ dB, and also because the symbolic calculation quickly becomes very difficult when the input is a sum of two sines with different frequencies.

III. NUMERICAL COMPARISONS

A. Equidistant points

It is not possible to make a formal comparison for different n because the resulting equations cannot be solved formally. What we therefore do in this section is to look at a relevant set of transfer characteristics and obtain numerical values for a given input signal $x = \sin \omega t$, i.e., $e = 1$.

As a set that should be representative for most real-world applications, we have chosen the following six transfer characteristics: T1 ($y = \tanh 4x$) simulates a heavily saturating differential pair and gives very strong odd-order distortion; T2 ($y = \tanh 1.3x$) simulates a saturating differential pair giving 10% H_2 and 1% H_4 ; T3 ($y = \tanh(0.36x + 0.06) - \tanh 0.06$) simulates a differential pair with a small input offset that has 1% H_2 and H_3 and 0.02% H_4 and H_5 ; T4 ($y = \tanh 0.11x$) simulates a differential pair with weak distortion, only 0.1% H_3 ; T5 ($y = e^x - 1$) simulates a one-transistor amplifier with very large distortion; and T6 ($y = \log(x+2) - \log 2$) simulates a logarithmic amplifier with very large distortion.²

²These functions are not plotted here because most look like straight lines to the human eye.

Extensive Matlab simulations for these (and also other functions) were made.³ The equidistant-point method was evaluated for $n = 3, 5, 7, 9$ points. To compare, the same simulations were made with using discrete Fourier transforms (DFTs) of length $n = 8, 10, 12$ where the sine period had to be chosen such that a rectangular window could be used. Finally, a 1024-point DFT simulation was made to calculate the exact value. The simulated values showed several things:

- For distortions of -30 dB or lower, the equidistant method with n points is at least as precise as a DFT with $2(n - 1)$ points.
- All distortions, including IM3, are computed with at least 0.1 dB accuracy, even if the lowest possible number of equidistant points⁴ is used, as long as the distortion values are -30 dB or lower. This is the first main result of this section: it shows a way to a very efficient simulation method discussed in the following section.
- For distortions of -20 dB, the second-lowest possible number of equidistant points needs to be used. For -10 dB of distortion, the equidistant-point does not give good results anymore.
- The equidistant-point method computes the distortion-induced offset very accurately (to within 1%) even if only three equidistant points are used. This is the second main result of this section: it shows how to derive simple formulae for distortion-induced offset.

B. Non-equidistant points

We also looked at the possibility of using non-equidistant points. For $n = 5$, instead of using $x = -2e, -e, 0, e, 2e$ to generate $y_1 \dots y_5$, one can use $x = -2e, -d, 0, d, 2e$ and ask what the optimum d is for estimating H_3 . The result: it depends on the transfer characteristic and on the level of H_3 . For the above transfer characteristics, $d = 0.99e \dots 1.02e$ is optimum for T3 to T6; for T2 it is $d = 0.95e$, for T1, $d = 0.85e$. So we can say that for harmonic distortions of -30 dB or below, using equidistant points is always an almost optimum choice.

IV. APPLICATIONS

A. Simulation of complex discrete-time amplifiers

The first application example was chosen to show how using the the equidistant-point method can speed up simulations. Hence the simulated circuit is a very complex switched-capacitor amplifier for a sensor read-out circuit. It consists of three stages: a pre-amplifier for the sensor signal, a configurable stage that can do simple amplification, chopping, or correlated double sampling (CDS), and an output buffer that drives the following ADC. The more than 20 switch control signals are generated by a timing-critical digital controller. The circuit has to be simulated with a mixed-analog-digital simulator.

³A table with detailed simulation results was omitted because it would not contain more information than described in this paper. However, the interested reader can request the table by e-mail from the author.

⁴This means: use the lowest odd n with which one can compute the harmonic distortion of the needed order.

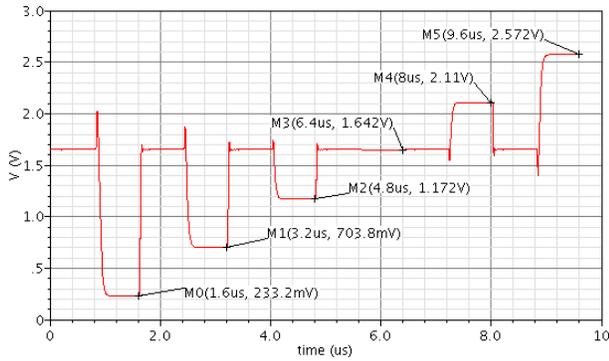


Fig. 2. Transient simulation of a complex switched-capacitor amplifier

Fig. 2 shows the output for a sensor signal that assumes the values $-3e$, $-2e$, $-e$, 0 , e , $2e$, one value every $1.6 \mu\text{s}$. The first of these cycles is not necessary for our method, but it ensures that the simulation is in a realistic state at the beginning of the second CDS cycle.⁵ The values at the end of the sampling period are then taken and inserted into (11) by rewriting (11) in the syntax of a simulator interface, e.g., for Cadence’s Analog Artist:

$$H2 = (20.0 * \log_{10}(\text{abs}(((-3.0 / 4.0) * ((\text{value}(\text{VT}("/\text{vout}")) 9.6\text{e-}06) - (2.0 * \text{value}(\text{VT}("/\text{vout}")) 6.4\text{e-}06))) + \text{value}(\text{VT}("/\text{vout}")) 3.2\text{e-}06))) / ((\text{value}(\text{VT}("/\text{vout}")) 9.6\text{e-}06) + \text{value}(\text{VT}("/\text{vout}")) 8\text{e-}06) - \text{value}(\text{VT}("/\text{vout}")) 4.8\text{e-}06) - \text{value}(\text{VT}("/\text{vout}")) 3.2\text{e-}06))))))$$

This example then gives a H_2 of -53.4 dB , a H_3 of -58.1 dB , and a distortion-induced offset of -8.16 mV .

Due to the complexity of the circuit, simulating just a single correlated-double-sampling cycle takes five minutes using Cadence’s ams/UltraSim on a 3-GHz dual-core processor. The six values shown in Fig. 2 took 30 minutes. This can be compared to the 55 minutes needed for a simulation with the 10-point DFT that gives a result of the same precision for these levels of distortion, or to the 5.4 hours needed if one follows the Cadence Analog Artist manual where a 64-point DFT is recommended to simulate distortion.

B. Monte-Carlo simulation of a discrete-time amplifier

The second application example was chosen to show how easily harmonic distortion Monte-Carlo or process corner simulations can be done. The circuit chosen for this is the discrete-time analog pad driver for microelectrode arrays presented as an application example in [7]. The big advantage of simulating harmonic distortion with equidistant input points is that the expressions are just rational functions of a few values. The simple expressions like the one of the first example can be directly used as output quantities in the simulator interface (for Cadence: the GUI Analog Artist or the script language Ocean).

The results are then immediately available to the tool—for Analog Artist, this means that the result is immediately

⁵This is needed because the simulator normally cannot make a correct DC convergence for switched-capacitor filters and will end up with some transistors in such strange states that the first simulated value cannot be trusted.

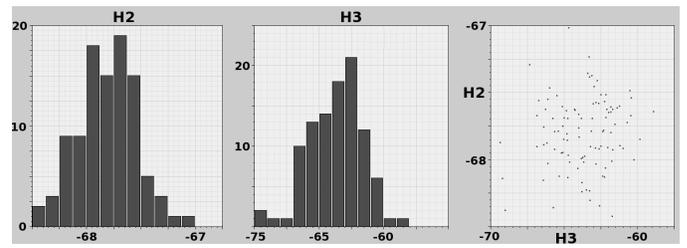


Fig. 3. Monte-Carlo simulation example. From left: Histograms for H_2 and H_3 in dB; scatter plot of H_2 and H_3 showing that they do not correlate well.

visible in the graphical user interface when the simulation terminates—which also makes it possible to easily evaluate distortion in the presence of element mismatch and process variations, parameter variations, and using Monte-Carlo simulations. This method was used in [7] to investigate into the H_2 , H_3 and the correlation between H_2 and H_3 for an output driver circuit; one resulting plot is shown in Fig. 3, where the distributions of H_2 and H_3 can be seen together with a scatter plot showing that for the amplifier in question, the two distortions are not correlated.

V. CALCULATING OFFSET AND DISTORTION

The equidistant-point method of calculating H_2 and H_3 from equidistant points on the transfer characteristic has already been used for calculating distortion, and the effects of mismatch on distortion [4], [5]. We can now also apply the method to calculating the distortion-induced offset, a topic that is, e.g., of interest when electromagnetic interference causes a signal outside the amplifier’s signal band to be present at an input pin. The result is then also useful to calculate how an amplitude-modulated signal will be demodulated by a non-linearity, since the frequency range of the signal modulated onto a carrier is like DC compared to the carrier frequency.⁶

As has been shown above, $n = 3$ is sufficient as long as the distortion does not exceed -30 dB . We just give a simple example with a single transistor. The current through a transistor in weak inversion is

$$I_d = I_m \exp\left(\frac{V_{gs} - V_m}{n\phi_t}\right), \quad (6)$$

where I_d is the drain current, V_{gs} the gate-source voltage, ϕ_t the thermal voltage, and I_m and V_m are process- and transistor-size-dependent quantities.

If we now have an input voltage $V_1(t)$ swinging around a bias voltage $V_{gs,bias}$, then the drain current becomes

$$I_d = I_m \exp\left(\frac{V_{gs,bias} - V_m}{n\phi_t}\right) \exp\left(\frac{V_1}{n\phi_t}\right). \quad (7)$$

The first two terms are signal independent, so we can collect them into a new quantity I'_m and write:

$$I_d = I'_m e^x \quad \text{with } x = \frac{V_1(t)}{n\phi_t}. \quad (8)$$

⁶One notable example for this would be the clicking noises that can be heard in many audio devices when a GSM phone is lying nearby.

Inserting this into (10) results in

$$I_{\text{off}} = I'_m \frac{\cosh x - 1}{2} \quad \text{with } x = \frac{V_1}{n\phi_t}, \quad (9)$$

where V_1 is the amplitude of the sine input $V_1(t)$. This result could now be used further to calculate the demodulation gain for an amplitude-modulated input signal.

VI. DISCUSSION AND CONCLUSION

The equidistant-point method discussed in this paper is very precise for the simulation and calculation of harmonic distortions below 3%, or -30 dB. Simulations are easy to set up and to evaluate with it. We have also shown that it requires less simulation time than the minimum-length simulation using a DFT for the calculation of the HD; for the most common case, where H_2 and H_3 are needed, the simulation time is only half as long than for the DFT method and even gives an estimate of IM3. The method's precision degrades quickly when harmonic distortions become stronger than 3%, which is normally not a problem, because very few systems tolerate more harmonic distortion.

The equidistant-point method works by measuring to what value a filter or amplifier settles for a specific input signal. This limits the application of the method to calculating the distortion in amplifiers and in the passband of low-pass filters (c.f. [6]).

In continuous-time systems, the method will only cover frequency-independent effects, not high-frequency effects that influence distortion. This means that if one wishes to apply the method to continuous-time systems, then one has to compare one or two simulations to a simulation using a sufficiently long DFT to make sure the dominant effects are quasi-stationary.

The situation is much better for switched-capacitor (SC) amplifiers and filters, because there even a single input value, when processed, is subject to all high-frequency effects. What the method will not cover is interference between successive samples in the SC amplifier, coming, for example, from kick-back noise into the voltage reference or from memory effects in the switches. Both are less pronounced in systems using auto-zeroing and correlated double sampling, but anyway, once it has been established that the reference is stable enough and that the memory effects are small enough, as it should be in SC filters, then the equidistant-point method can be used for corner simulations and Monte-Carlo simulations that would otherwise be very time-consuming.

This method is used in our group to cut down on simulation time, although the effect we observe is that using this method, we do not simulate less, but make simulations like Monte-Carlo harmonic-distortion simulations that we would otherwise not make because they would simply take too long.

VII. APPENDIX

$$n = 3: \quad \text{Offset} = \frac{y_1 - 2y_2 + y_3}{4} \quad (10)$$

$$H_2 = \frac{y_1 - 2y_2 + y_3}{2y_1 - 2y_3}$$

$$n = 5: \quad \text{Offset} = \frac{2y_1 + 4y_2 - 12y_3 + 4y_4 + 2y_5}{12}$$

$$H_2 = \frac{3(y_1 - 2y_3 + y_5)}{4(y_1 + y_2 - y_4 - y_5)} \quad (11)$$

$$H_3 = \frac{-y_1 + 2y_2 - 2y_4 + y_5}{2(y_1 + y_2 - y_4 - y_5)}$$

$$H_4 = \frac{-y_1 + 4y_2 - 6y_3 + 4y_4 - y_5}{4(y_1 + y_2 - y_4 - y_5)}$$

$$\text{IM3} = \frac{3(y_1 - 2y_2 + 2y_4 - y_5)}{2(4y_1 - 5y_2 + 5y_4 - 4y_5)}$$

$$n = 7: \quad \text{Offset} = \frac{167y_1 + 378y_2 - 135y_3 - 820y_4 - 135y_5 + 378y_6 + 167y_7}{1280}$$

$$H_2 = \frac{559y_1 + 486y_2 - 1215y_3 + 340y_4 - 1215y_5 + 486y_6 + 559y_7}{4(167y_1 + 252y_2 - 45y_3 + 45y_5 - 252y_6 - 167y_7)}$$

$$H_3 = \frac{-45(5y_1 - 4y_2 - 7y_3 + 7y_5 + 4y_6 - 5y_7)}{2(167y_1 + 252y_2 - 45y_3 + 45y_5 - 252y_6 - 167y_7)}$$

$$H_4 = \frac{-9(17y_1 - 42y_2 + 15y_3 + 20y_4 + 15y_5 - 42y_6 + 17y_7)}{2(167y_1 + 252y_2 - 45y_3 + 45y_5 - 252y_6 - 167y_7)}$$

$$H_5 = \frac{81(y_1 - 4y_2 + 5y_3 - 5y_5 + 4y_6 - y_7)}{2(167y_1 + 252y_2 - 45y_3 + 45y_5 - 252y_6 - 167y_7)}$$

$$H_6 = \frac{81(y_1 - 6y_2 + 15y_3 - 20y_4 + 15y_5 - 6y_6 + y_7)}{4(167y_1 + 252y_2 - 45y_3 + 45y_5 - 252y_6 - 167y_7)}$$

$$n = 9: \quad \text{Offset} = \frac{69y_1 + 176y_2 - 126y_3 + 336y_4 - 910y_5 + 336y_6 - 126y_7 + 176y_8 + 69y_9}{630}$$

$$H_2 = \frac{243y_1 + 352y_2 - 672y_3 + 672y_4 - 1190y_5 + 672y_6 - 672y_7 + 352y_8 + 243y_9}{12(23y_1 + 44y_2 - 21y_3 + 28y_4 - 28y_6 + 21y_7 - 44y_8 - 23y_9)}$$

$$H_3 = \frac{-35(y_1 - 2y_3 + 2y_7 - y_9)}{2(23y_1 + 44y_2 - 21y_3 + 28y_4 - 28y_6 + 21y_7 - 44y_8 - 23y_9)}$$

$$H_4 = \frac{-7(23y_1 - 32y_2 - 28y_3 + 32y_4 + 10y_5 + 32y_6 - 28y_7 - 32y_8 + 23y_9)}{12(23y_1 + 44y_2 - 21y_3 + 28y_4 - 28y_6 + 21y_7 - 44y_8 - 23y_9)}$$

$$H_5 = \frac{28(y_1 - 3y_2 + 2y_3 + y_4 - y_6 - 2y_7 + 3y_8 - y_9)}{3(23y_1 + 44y_2 - 21y_3 + 28y_4 - 28y_6 + 21y_7 - 44y_8 - 23y_9)}$$

$$H_6 = \frac{2(9y_1 - 44y_2 + 84y_3 - 84y_4 + 70y_5 - 84y_6 + 84y_7 - 44y_8 + 9y_9)}{3(23y_1 + 44y_2 - 21y_3 + 28y_4 - 28y_6 + 21y_7 - 44y_8 - 23y_9)}$$

$$H_7 = \frac{-8(y_1 - 6y_2 + 14y_3 - 14y_4 + 14y_6 - 14y_7 + 6y_8 - y_9)}{3(23y_1 + 44y_2 - 21y_3 + 28y_4 - 28y_6 + 21y_7 - 44y_8 - 23y_9)}$$

$$H_8 = \frac{-4(y_1 - 8y_2 + 28y_3 - 56y_4 + 70y_5 - 56y_6 + 28y_7 - 8y_8 + y_9)}{3(23y_1 + 44y_2 - 21y_3 + 28y_4 - 28y_6 + 21y_7 - 44y_8 - 23y_9)}$$

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