Minimum-Sensitivity Single-Amplifier Biquadratic Filters

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Abstract

Evaluating the Schoeffler criterion for Sallen-Key filters leads to expressions having two degrees of freedom and prohibitive complexity. Such expressions are normally solved only after making approximations. We show, using the low-pass filter as an example, how the two degrees of freedom can be separated by a simple non-linear coordinate transform. Exact design equations for the minimum-sensitivity filter result, revealing that this filter has the maximum allowable component spread, that its capacitor spread is larger than the resistor spread, and that the required amplifier gain does not exceed two. We then repeat the analysis for unity-gain filters and show that the minimum-sensitivity filter has *minimum* component spreads if its pole frequency is below a certain value.

1 Introduction

Discrete-component filters are often implemented as cascades of single-amplifier biquadratic filters (SABs), because they are cheaper than multi-amplifier filters. In IC form, SABs are also preferred, because they consume less power than their multi-amplifier counterparts (cf. [1]). The disadvantage of SABs is the comparatively high sensitivity of the pole quality factor q_p to variations of the passive component values and of the amplifier gain. Design equations for minimizing this sensitivity are well known (cf. [2, 3]), but they were all derived by first making approximations and then solving for a minimum-sensitivity filter.

In this paper, we solve the optimization problem without making any approximations. First, we derive design equations for the Sallen-Key lowpass filter using a non-linear coordinate transform. We then prove some general properties of the minimum-sensitivity filter: First, it will have the maximum allowable component spread; second, the capacitor spread will be larger than the resistor spread; and last, but not least, we show that the gain of the minimum-sensitivity filter is less than two.

A similar discussion for the unity-gain lowpass filter provides an even more interesting result: if the pole frequency is below a certain limit determined by the amplifier, the resistors' variance, and q_p, then the minimum-sensitivity filter has *very low* component spreads. Although this will not happen for high-Q filters with a pole frequency pushed to the physical limits, many low-Q anti-aliasing filters be built with very low component spreads.

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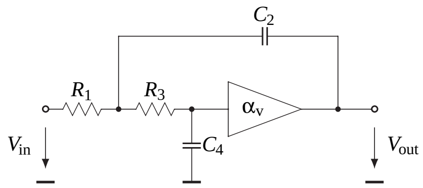

Figure 1: Biquadratic Sallen-Key lowpass filter.

2 Arbitrary-Gain Sallen-Key Lowpass Filter

Figure 1 shows a second-order Sallen-Key lowpass filter. To simplify the arithmetic, its passive components can be written as $R_1 = R/n$, $R_3 = R \cdot n$, $C_2 = C/m$ and $C_4 = C \cdot m$. Then R and C are the geometric means of R_1, R_3 and C_2, C_4, n and m are the *component spread factors* of the resistors and the capacitors, and the component spreads are $\max\{n^2, 1/n^2\}$ and $\max\{m^2, 1/m^2\}$. The filter's transfer function is

$$T(s) = \alpha_V \frac{\omega_\text{p}^2}{s^2 + \frac{\omega_\text{p}}{q_\text{p}} s + \omega_\text{p}^2} ,$$

$$\text{with} \quad \omega_\text{p} = \frac{1}{RC} , \quad \frac{1}{q_\text{p}} = mn + \frac{m}{n} + \frac{1-\alpha_V}{mn} , \quad (1)$$

where ω_p is the pole frequency in rad/s and q_p is the pole quality factor. They have the following active and passive sensitivities:

$$S_{\alpha_\text{v}}^{\omega_\text{p}} = 0 , \quad S_{\alpha_\text{v}}^{q_\text{p}} = 2\alpha_\text{v}/D , \quad S_{R_{1,3},C_{2,4}}^{\omega_\text{p}} = -1/2 , \quad (2a)$$

$$S_{R_1}^{q_\text{p}} = -S_{R_3}^{q_\text{p}} = \left(m^2n^2 - m^2 - 1 + \alpha_\text{v}\right)/D , \quad (2b)$$

$$S_{C_2}^{q_\text{p}} = -S_{C_4}^{q_\text{p}} = \left(m^2n^2 + m^2 - 1 + \alpha_\text{v}\right)/D , \quad (2c)$$

$$\text{where} \quad D = 2 \cdot \left(m^2n^2 + m^2 + 1 - \alpha_\text{v}\right) .$$

It becomes apparent from (2a)–(2c) that, using resistors and capacitors of a given precision, only the sensitivities of q_p differ for different designs. Therefore, all that can be done is to minimize the variance of q_p in function of variances of the component values. The expressions become even simpler if *relative* variances are used, e.g. $\overline{\sigma}_{R_1}^2 = \sigma_{R_1}^2 / R_1^2$. Then

$$\overline{\sigma}_{q_\text{p}}^2 \approx \sum_{x \in X} \left(S_x^{q_\text{p}}\right)^2 \overline{\sigma}_x^2 , \quad X = \{R_1, C_2, R_3, C_4, \alpha_\text{v}\} , \quad (3)$$

which is called *Schoeffler's multivariate criterion*. Substituting (2a)–(2c) into this equation results in a closed-form expression for $\overline{\sigma}_{q_\text{p}}^2$. Finding all local minima now means setting the gradient to zero, i.e. solving the vector equation $\nabla \overline{\sigma}_{q_\text{p}}^2 = \mathbf{0}$ for m and n, which cannot be done directly.

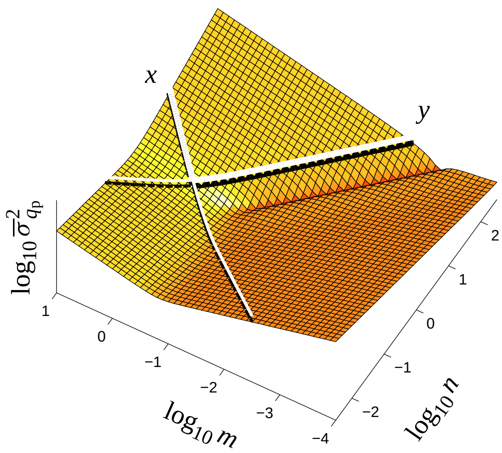

Figure 2: Example of $\overline{\sigma}^2_{q_\text{p}}$ with the x, y coordinate system shown by white lines.

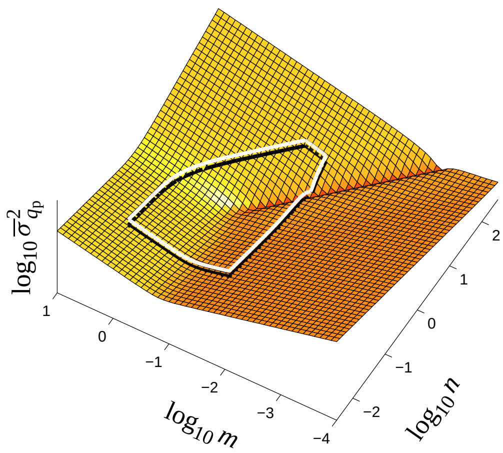

Figure 3: Surface from Fig. 2 with the boundary described by $\alpha_\text{v} < 5$, Component spread < 1000.

Visualizing numerical examples of the functions (3) around $m = n = 1$ shows, however, that they all have a form similar to the one shown in Fig. 2. A "valley" towards the right is apparent, which can be brought into the direction of an axis using a simple non-linear coordinate transformation,

$$x = mn, \quad y = \frac{n}{m}, \quad \text{with } m, n, x, y > 0\,, \tag{4}$$

which is just a 45-degree rotation of the logarithmic coordinate system. The new coordinates are shown as white lines in Figure 2.

Solving $\nabla\overline{\sigma}^2_{q_\text{p}} = \mathbf{0}$ in the new coordinates is an almost trivial task. We give the results in implicit form:

$$0 = 2q_\text{p}\overline{\sigma}^2_R(2\overline{\sigma}^2_C + \overline{\sigma}^2_{\alpha_\text{v}})\hat{x}^4 - \overline{\sigma}^2_R(2\overline{\sigma}^2_C + \overline{\sigma}^2_{\alpha_\text{v}})\hat{x}^3 + \overline{\sigma}^2_C\overline{\sigma}^2_{\alpha_\text{v}}\hat{x} - 2\overline{\sigma}^2_C\overline{\sigma}^2_{\alpha_\text{v}} q_\text{p}\,, \tag{5a}$$

$$\hat{y} = \frac{-q_\text{p}\hat{x}(2\overline{\sigma}^2_C + \overline{\sigma}^2_{\alpha_\text{v}})}{q_\text{p}(2\overline{\sigma}^2_C + \overline{\sigma}^2_{\alpha_\text{v}})\hat{x}^2 - (\overline{\sigma}^2_C + \overline{\sigma}^2_{\alpha_\text{v}})\hat{x} + \overline{\sigma}^2_{\alpha_\text{v}} q_\text{p}}\,. \tag{5b}$$

We show in the Appendix that this system of equations has no solution for $q_\text{p} > \frac{1}{2}$ with $\hat{x}, \hat{y} > 0$. Thus the minimum-sensitivity solution lies on the boundary defined by the maximum allowable component spreads (see Fig. 3). The "bottom of the valley" in Fig. 2 is at $\hat{x}_\infty$ (see Fig. 4), given by

$$q_\text{p}(2\overline{\sigma}^2_R + 2\overline{\sigma}^2_C + \overline{\sigma}^2_{\alpha_\text{v}})\hat{x}^4_\infty - (\overline{\sigma}^2_R + \overline{\sigma}^2_C + \overline{\sigma}^2_{\alpha_\text{v}})\hat{x}^3_\infty + \overline{\sigma}^2_{\alpha_\text{v}}\hat{x}_\infty - q_\text{p}\overline{\sigma}^2_{\alpha_\text{v}} = 0\,. \tag{6}$$

This equation was derived from (5a)–(5b) for $y \to \infty$. It can be used as a design equation, provided that the allowable component spread is large enough ($1/m^2 \gtrsim 10$ in our example). Note that $\hat{x}_\infty < 1$ for $m < 1$, which means that the resistor spread is always smaller than the capacitor spread. Finally, the gain α_v at the bottom of the valley is always less than 2, which will also be shown in the Appendix. This is an advantage, since

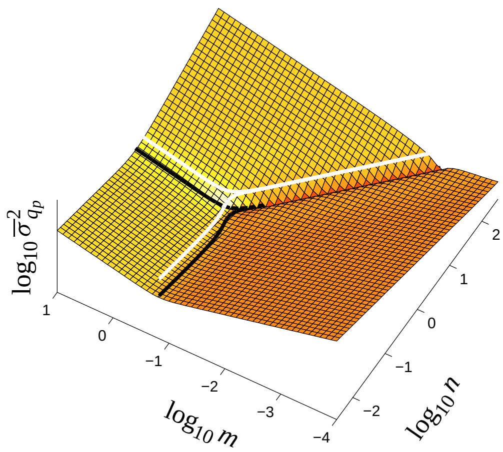

Figure 4: Example of $\overline{\sigma}^2_{q_p}$ showing white lines above the surface where $\nabla_m\overline{\sigma}^2_{q_p} = 0$ and $\nabla_n\overline{\sigma}^2_{q_p} = 0$.

the higher the gain of a voltage amplifier built using an opamp, the higher its noise and distortion, and the lower its bandwidth.

3 Unity-Gain Sallen-Key Lowpass Filter

Sallen-Key filters are often built around an opamp connected as a unity-gain buffer. In this case, (1) can be solved for m:

$$m = \frac{n}{q_\text{p}(n^2 + 1)}\,. \tag{7}$$

It can be seen that $m \geq q_\text{p}/2$ for all possible n, which is a well-known result (cf. [2]). For complex poles ($q_\text{p} > 0.5$), $m < 1$ and therefore $C_4 < C_2$. In other words, given a maximum allowable capacitor spread of $1/m^2$, only filters with $q_\text{p} \leq \frac{1}{2} m$ can be built.

There remains only one degree of freedom, and the minimum-sensitivity filter can be found by solving

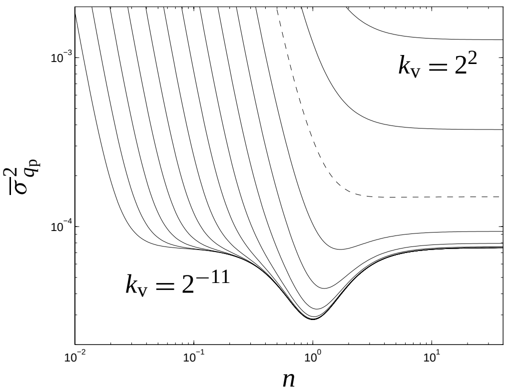

Figure 5: $\overline{\sigma}^2_{q_\mathrm{p}}$ for different values of k_v (dashed: $k_\mathrm{v} = 1$)

$\nabla\sigma^2_{q_p} = 0$, which leads to the equation

$$4\left(\overline{\sigma}^2_{\alpha_\mathrm{v}} q_\mathrm{p}^4 - \overline{\sigma}^2_R\right) n^8 + 4\left(4\overline{\sigma}^2_{\alpha_\mathrm{v}} q_\mathrm{p}^4 + \overline{\sigma}^2_R\right) n^6 + 24\overline{\sigma}^2_{\alpha_\mathrm{v}} q_\mathrm{p}^4 n^4 + 16\overline{\sigma}^2_{\alpha_\mathrm{v}} q_\mathrm{p}^4 n^2 + 4\overline{\sigma}^2_{\alpha_\mathrm{v}} q_\mathrm{p}^4 = 0$$

under the condition that $\overline{\sigma}^2_R$ and $\overline{\sigma}^2_C$ are identical for all resistors and capacitors, respectively. Note that the solution is independent of $\overline{\sigma}^2_C$. After the substitution $\overline{\sigma}^2_{\alpha_\mathrm{v}} = \overline{\sigma}^2_R k_\mathrm{v}^2 / q_\mathrm{p}^4$, this equation becomes

$$\left(n^2+1\right)^4 k_\mathrm{v}^2 - n^6\left(n^2-1\right) = 0\,. \qquad (8)$$

It can easily be seen that (8) can have one real root in the range $1 \le n < \infty$, corresponding to $0 \le k_\mathrm{v}^2 < 1$ or

$$q_\mathrm{p}^2 < \frac{\overline{\sigma}_R}{\overline{\sigma}_{\alpha_\mathrm{v}}}\,. \qquad (9)$$

If condition (9) is not met, the minimum-sensitivity filter is again defined by the maximum allowable component spread, and the minimum achievable variance becomes $\frac{1}{2} q_\mathrm{p}^2 (\overline{\sigma}^2_R + \overline{\sigma}^2_C + 2\overline{\sigma}^2_{\alpha_\mathrm{v}} q_\mathrm{p}^4)$.

As an example, Fig. 5 shows $\overline{\sigma}^2_{q_\mathrm{p}}$ for $\overline{\sigma}_R = \overline{\sigma}_C = 1\%$, $q_\mathrm{p} = 3$ and $k_\mathrm{v} = 2^{-11} \ldots 2^2$. It is apparent that the minimum normally occurs at $n \approx 1$, i.e. for a very small resistor spread, where $\overline{\sigma}^2_{q_\mathrm{p}}$ is much smaller than it is for $n \gg 1$. It also appears that not much is to be gained if k_v is brought below 1/8.

To illustrate this, two numerical examples follow. Consider a lowpass filter with $q_\mathrm{p} = 2$. Then a minimum exists for $\overline{\sigma}_{\alpha_\mathrm{v}} < \frac{1}{4}\overline{\sigma}_R$. If 1 % resistors are used, the gain must not vary by more than 0.25 % (i.e. $\overline{\sigma}_{\alpha_\mathrm{v}} < 0.0025$). The opamp's open-loop gain is approximately GBW/f, and therefore

$$\overline{\sigma}_{\alpha_\mathrm{v}} \approx \frac{f}{GBW}\overline{\sigma}_{GBW}\,,$$

where GBW is the Gain-Bandwidth Product of the opamp used to build the buffer. With $GBW = 1.5\,\mathrm{MHz}$ and $\overline{\sigma}_{GBW} \approx 50\,\%$,[1] the filter should be built with low component spreads if it has a pole frequency f_p below 7.5 kHz. For a Butterworth or "maximally flat" filter (often used for anti-aliasing), $q_\mathrm{p} = 1/\sqrt{2}$. Therefore $\overline{\sigma}_{\alpha_\mathrm{v}} < 2$ %, and a minimum exists for all $f_\mathrm{p} < 60\,\mathrm{kHz}$.

4 Conclusion

In this paper, we have discussed minimum-sensitivity Sallen-Key lowpass filters, and have demonstrated that their gain is always smaller than two. Some of the equations presented in this paper[2] can also be used to design a minimum-sensitivity filter with given pole frequency, pole quality factor, and component tolerances.

The same analysis can also be carried out for other Sallen-Key filters, giving similar results. The highpass filter is dual to the lowpass filter, which means that the resistors and capacitors must be interchanged; thus the same results are valid. The results are slightly different for the two dual bandpass filters, in which case the gain of the minimum-sensitivity filter is upper-bounded by four instead of two.

5 Appendix

Proof for the statement that Equations (5a)–(5b) **have no solution for $x > 0$, $y > 0$, $q_\mathrm{p} > \frac{1}{2}$.** Outline:

1. Calculate the root locus of (5a) for $0 \le q_\mathrm{p} \le \infty$ and show that it has exactly one real root if $q_\mathrm{p} > \frac{1}{2}$.
2. Express q_p as a function of $\hat{x}$ and show that $\hat{y}$ is negative over the whole range of $\hat{x}$ for $\frac{1}{2} < q_\mathrm{p} < \infty$.

Part 1 — Rewrite (5a) as a polynomial in q_p:

$$q_\mathrm{p}\left(2\overline{\sigma}^2_R(2\overline{\sigma}^2_C + \overline{\sigma}^2_{\alpha_\mathrm{v}})\hat{x}^4 - 2\overline{\sigma}^2_C\overline{\sigma}^2_{\alpha_\mathrm{v}}\right) - \left(\overline{\sigma}^2_R(2\overline{\sigma}^2_C + \overline{\sigma}^2_{\alpha_\mathrm{v}})\hat{x}^3 - \overline{\sigma}^2_C\overline{\sigma}^2_{\alpha_\mathrm{v}}\hat{x}\right) = 0\,. \qquad (10)$$

The four roots can be calculated at three special points:

q_p	Roots	(Finite) Positive Real Root
0	$0, (k_\sigma)^{\frac{1}{2}}, \infty$	$\hat{x}\vert_{q_\mathrm{p}=0} = \sqrt{k_\sigma}$
$\frac{1}{2}$	$1, (-k_\sigma)^{\frac{1}{3}}$	$\hat{x}\vert_{q_\mathrm{p}=\frac{1}{2}} = 1$
∞	$(k_\sigma)^{\frac{1}{4}}$	$\hat{x}\vert_{q_\mathrm{p}=\infty} = \sqrt[4]{k_\sigma}$

$$\text{with} \qquad k_\sigma = \frac{\overline{\sigma}^2_C\overline{\sigma}^2_{\alpha_\mathrm{v}}}{\overline{\sigma}^2_R(2\overline{\sigma}^2_C + \overline{\sigma}^2_{\alpha_\mathrm{v}})}\,.$$

[1] This approximately describes the well-known LM 741.
[2] i.e. (1), (4), (6), (7), (8) and (9)

There are only two fundamentally different root locii:

condition	$\hat{x}$ for $\infty > q_{\mathrm{p}} > \frac{1}{2}$
$k_\sigma < 1$	$\sqrt[4]{k_\sigma} < \hat{x} < 1$
$k_\sigma > 1$	$\sqrt[4]{k_\sigma} > \hat{x} > 1$

One example of each is shown in Fig. 6. This shows that, in both cases, there is exactly one positive real root for $q_{\mathrm{p}} > \frac{1}{2}$.

Part 2: — Express q_{p} as a function of x:

$$q_{\mathrm{p}} = \frac{\overline{\sigma}_R^2(2\overline{\sigma}_C^2 + \overline{\sigma}_{\alpha_{\mathrm{v}}}^2)x^3 + \overline{\sigma}_C^2\overline{\sigma}_{\alpha_{\mathrm{v}}}^2 x}{2\overline{\sigma}_R^2(2\overline{\sigma}_C^2 + \overline{\sigma}_{\alpha_{\mathrm{v}}}^2)x^4 - 2\overline{\sigma}_C^2\overline{\sigma}_{\alpha_{\mathrm{v}}}^2} . \qquad (11)$$

Substitute (11) into equation (5b):

$$\hat{y} = \frac{x}{\overline{\sigma}_{\alpha_{\mathrm{v}}}^2} \cdot \frac{\overline{\sigma}_R^2(2\overline{\sigma}_C^2 + \overline{\sigma}_{\alpha_{\mathrm{v}}}^2)x^2 - \overline{\sigma}_C^2\overline{\sigma}_{\alpha_{\mathrm{v}}}^2}{\overline{\sigma}_R^2 x^4 + (\overline{\sigma}_C^2 - \overline{\sigma}_R^2)x^2 - \overline{\sigma}_C^2} .$$

Both numerator and denominator have exactly one positive real root, namely $\sqrt{k_\sigma}$ and 1. There are two possibilities:

k_σ < 1: $\hat{y} < 0$ for $\sqrt{k_\sigma} < \hat{x} < 1$. Since $\sqrt{k_\sigma} < \sqrt[4]{k_\sigma}$, this range contains all $\hat{x}$ for $\frac{1}{2} < q_{\mathrm{p}} < \infty$, and there is no solution to the optimization problem.

k_σ > 1: $\hat{y} < 0$ for $\sqrt{k_\sigma} > \hat{x} > 1$. For the same reason as before, there is no solution to the optimization problem.

Proof for the statement that the gain of the minimum-sensitivity filter does not exceed 2.

Figure 4 shows the same example as Fig. 2 including the lines on which the gradients in the directions of the m and n axes are zero. For $y \gg 1$, these lines converge towards the same x-value $\hat{x}_\infty$, which is described by (6) above. As before, a root locus analysis can be made:

$$\hat{x}_\infty|_{q_{\mathrm{p}}=0} = \sqrt{k_{\sigma 0}} \qquad k_{\sigma 0} = \frac{\overline{\sigma}_{\alpha_{\mathrm{v}}}^2}{\overline{\sigma}_R^2 + \overline{\sigma}_C^2 + \overline{\sigma}_{\alpha_{\mathrm{v}}}^2}$$

$$\hat{x}_\infty|_{q_{\mathrm{p}}=\frac{1}{2}} = 1$$

$$\hat{x}_\infty|_{q_{\mathrm{p}}=\infty} = \sqrt[4]{k_{\sigma\infty}} \qquad k_{\sigma\infty} = \frac{\overline{\sigma}_{\alpha_{\mathrm{v}}}^2}{2\overline{\sigma}_R^2 + 2\overline{\sigma}_C^2 + \overline{\sigma}_{\alpha_{\mathrm{v}}}^2}$$

This time, there is only one type of root locus, since $\sqrt{k_{\sigma 0}} < \sqrt[4]{k_{\sigma\infty}} < 1$ for all choices of $\overline{\sigma}_R^2, \overline{\sigma}_C^2, \overline{\sigma}_{\alpha_{\mathrm{v}}}^2$. It has the same form as the one at the top of Fig. 6. Therefore it follows that

$$\left(\frac{\overline{\sigma}_{\alpha_{\mathrm{v}}}^2}{2\overline{\sigma}_R^2 + 2\overline{\sigma}_C^2 + \overline{\sigma}_{\alpha_{\mathrm{v}}}^2}\right)^{\frac{1}{4}} < \hat{x}_\infty < 1 \quad \text{for } \infty > q_{\mathrm{p}} > \frac{1}{2} ,$$

and $\hat{x}_\infty$ is always smaller than unity. It follows from equation (4) that the sensitivity is minimum at an $n < \frac{1}{m}$ for large n. Thus the filter should have high component spreads, where the capacitor spread is always greater (but normally not much greater) than the resistor spread.

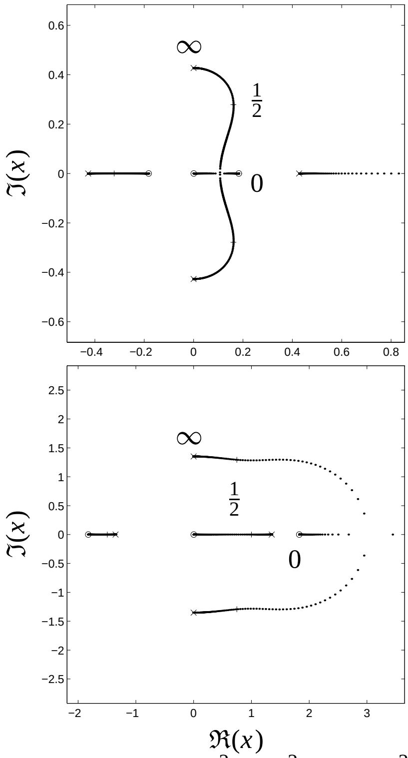

Figure 6: Root locii for $\overline{\sigma}_C^2 = \overline{\sigma}_{\alpha_{\mathrm{v}}}^2 = 0.1\overline{\sigma}_R^2$ (top) and $\overline{\sigma}_C^2 = \overline{\sigma}_{\alpha_{\mathrm{v}}}^2 = 10\overline{\sigma}_R^2$ (bottom). '∘' are the roots for $q_{\mathrm{p}} = 0$, '+' for $q_{\mathrm{p}} = \frac{1}{2}$ and '×' for $q_{\mathrm{p}} \to \infty$

Equation (1) is now translated into x, y coordinates, and the limit $y \gg 1$ is taken. Solved for q_{p}:

$$q_{\mathrm{p}} = \frac{x}{x^2 + 1 - \alpha_{\mathrm{v}}} . \qquad (12)$$

Substitute (12) into (6), and solve for α_{v}:

$$\alpha_{\mathrm{v}} = -x^2 \frac{(\overline{\sigma}_R^2 + \overline{\sigma}_C^2)x^2 - (\overline{\sigma}_R^2 + \overline{\sigma}_C^2)}{(\overline{\sigma}_R^2 + \overline{\sigma}_C^2 + \overline{\sigma}_{\alpha_{\mathrm{v}}}^2)x^2 - \overline{\sigma}_{\alpha_{\mathrm{v}}}^2} .$$

This function has four zeros $(-1, 0, 0, 1)$ and two poles $\left(\pm\sqrt{k_{\sigma 0}}\right)$, and thus is positive over the whole range of $\hat{x}$. It is also monotonically increasing from $\alpha_{\mathrm{v}} = 0$ for $q_{\mathrm{p}} = \frac{1}{2}$ up to $\alpha_{\mathrm{v}} = 1 + \sqrt{k_{\sigma\infty}}$ for $q_{\mathrm{p}} \to \infty$, and can therefore not be greater than 2.

References

[1] Hanspeter Schmid and George S. Moschytz, "A tunable, video-frequency, low-power, single-amplifier biquadratic filter in CMOS," in *Proc. ISCAS*, Orlando, Florida, June 1999.

[2] Lawrence P. Huelsman and Phillip E. Allen, *Introduction to the theory and design of active filters*, McGraw-Hill Book Company, New York, 1980.

[3] George S. Moschytz and P. Horn, *Active filter design handbook*, John Wiley & Sons, Chichester, 1981.